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TRANSIENT ENERGY TRANSFER BY RADIATION AND CONDUCTION

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Abstract—The transient energy transfer by radiation and conduction through a semi-infinite medium has been investigated. A kernel substitution technique has been used in order to obtain analytic solutions and hence to display readily the main features and parameters of the problem. The temperature and heat flux as a function of position and time are shown and compared with the pure conduction solution.

NOTATION

a, speed of propagation of thermal wave;
$a_{1,2}$, dimensionless parameters, defined by equations (25) and (26);
c, specific heat;
D, dimensionless parameter, defined by equation (27);
E_n, exponential integral, $E_n = \int_0^1 \mu^{n-2} \exp [- t/\mu]\, d\mu$;
k, radiative absorption coefficient;
q, dimensionless heat flux, $q = q^*/\sigma T_0^4$;
q^*, heat flux, energy/area-time;
t, dimensionless time, $t = 3k\sigma T_0^3 t^*/\rho c$;
t^*, time;
T, dimensionless perturbation temperature, $T = T^\dagger - 1$;
$T^\dagger$, dimensionless temperature, $T^\dagger = T^*/T_0$;
T, temperature;
T_0, initial temperature of medium;
T_r, effective temperature of external radiation;
T_s, surface temperature;
x, distance.

Greek symbols

$\gamma_{1,2}$, dimensionless parameters, defined by equation (17);
$\delta_{1,3}$, dimensionless parameters, defined by equations (11) and (24);
λ, thermal conductivity;
ρ, density;
σ, Stefan–Boltzmann constant;
τ^*, optical depth, $\tau^* = kx$;
τ, modified optical depth, $\tau = 3\tau^*/2$.

INTRODUCTION

THE SOLUTION to the problem of transient energy transfer by pure conduction in a semi-infinite medium is well known. The analogous problem with energy transfer by both radiation and conduction is of importance both because of its fundamental nature and because of its practical aspects. Transient energy transfer by radiation and conduction is of interest, for instance, in astrophysical problems and in re-entry heating problems.

The related problem of energy transfer by radiation and convection has been studied extensively (see [8], [9], and the authors referred to in these papers). After this work was completed, it was pointed out to the author that Nemchinov [9] had previously obtained a first approximation to the linearized problem of transient energy transfer by radiation and conduction. The present work approximates the solution more accurately and includes more general boundary conditions. The higher approximations show the wave-like character of the temperature field when the radiative energy is dominant as compared with conductive energy transfer. The effects of external radiation and surface temperature changes are also studied. Although the linear approximation is retained, important features such as position and approximate magnitude of rapid temperature changes are obtained which will still be present in the non-linear problem.

The present analysis employs a kernel substitution technique which has previously been shown to be quite accurate [1, 2, 8]. The approximations involved in this technique are discussed in [1, 2]. The kernel substitution method allows the basic integro-differential equation governing the energy transfer, either linear or non-linear, to be reduced to a differential equation. The method is applicable to other problems governed by the interaction of radiation, conduction, and convection both steady and time-dependent. This method and related techniques which also have reduced integro-differential equations to differential equations, have previously been applied to radiative transfer problems by several authors of whom the first were probably Schuster [3] and Schwarzschild [4]. Chandrasekhar [5] should be consulted for a discussion of the various methods employed.

BASIC EQUATIONS

Statement of the problem

Consider a semi-infinite medium capable of transferring energy by radiation and conduction. Initially the medium is at a uniform temperature T_0 throughout. For time $t^* > 0$, the surface temperature T^* is maintained at T_s^*, a constant, and a constant radiant energy q_s^* is incident on the surface. The temperature and energy flux as a function of position and time are required.

The equation governing the time-dependent energy flux is:

$$-\frac{\partial q^*}{\partial x} = \rho c \frac{\partial T^*}{\partial t^*} \tag{1}$$

Distance x is measured from the surface towards the interior of the medium. ρ is the density and c is the specific heat of the medium. The total energy flux q^* is the sum of the conductive heat flux q_c^* and the radiative heat flux q_r^*.

The conductive heat flux is given by:

$$q_c^* = -\lambda \frac{\partial T^*}{\partial x} \tag{2}$$

where λ is the thermal conductivity. If it is assumed that the absorption coefficient k is independent of frequency and temperature, the radiative heat flux is given by [2, 5]

$$q_r^* = 2\sigma \int_0^{\tau^*} T^{*4}(\bar{\tau}^*)\, E_2(\tau^* - \bar{\tau}^*)\, d\bar{\tau}^* - 2\sigma \int_{\tau^*}^{\infty} T^{*4}(\bar{\tau}^*)\, E_2(\bar{\tau}^* - \tau^*)\, d\bar{\tau}^* + 2\, q_s^*\, E_3(\tau^*) \tag{3}$$

σ is the Stefan–Boltzmann constant. E_n is the exponential integral defined as

$$E_n(\tau^*) = \int_0^1 \mu^{n-2} \exp(-\tau^*/\mu)\, d\mu \tag{4}$$

and τ^* is the optical depth, $\tau^* = kx$.

It has been implicitly assumed in writing the last term in equation (3) that the external radiation q_s is diffuse, and also that the surface is completely transparent to radiation.

Kernel substitution

Equations (1)–(3) with the appropriate boundary conditions completely specify the problem and a solution may be obtained by numerical analysis. However, to obtain an analytic solution, it is convenient to substitute in equation (3) an approximate kernel of the form $a \exp(-bx)$ for the correct kernel $E_2(x)$ and also $a \exp(-bx)/b$ for the kernel $E_3(x)$. The constants a and b are determined by requiring the exponential kernel duplicate the main features of the exponential integral kernel, i.e. by requiring that the area and first moment of the exponential kernel be equal to the area and first moment of the exponential integral kernel. It is found that $a = 3/4$ and $b = 3/2$.

By means of this substitution and by introducing the dimensionless variables

$$q = \frac{q^*}{\sigma T_0^4},\ T^\dagger = \frac{T^*}{T_0},\ \tau = \frac{3}{2}\tau^* \tag{5}$$

the equation for the radiative flux becomes

$$q_r = \int_0^{\tau} T^{\dagger 4} \exp[-(\tau - \bar{\tau})]\, d\bar{\tau} - \int_{\tau}^{\infty} T^{\dagger 4} \exp[-(\bar{\tau} - \tau)]\, d\bar{\tau} + q_s \exp(-\tau) \tag{6}$$

Linearization

Equation (6) may be further simplified by the process of linearization without however eliminating any of the overall characteristics of the problem. If we confine our attention to phenomena such that

$$T^{\dagger} = 1 + T \tag{7}$$

where T is a small perturbation, then the radiative heat flux becomes

$$q_r = 4 \int_{\tau}^{\infty} T \exp[-(\tau - \bar{\tau})] \, d\bar{\tau} - 4 \int_{\tau}^{\infty} T \exp[-(\bar{\tau} - \tau)] \, d\bar{\tau} + 4T_r \exp(-\tau) \tag{8}$$

The temperature T_r is given by

$$T_r = \frac{q_s - 1}{4} \tag{9}$$

and defines an effective temperature of the external radiation.

By substituting equations (8) and (2) into equation (1), one obtains

$$\delta_1 \frac{\partial^2 T}{\partial \tau^2} - \frac{\partial T}{\partial t} = -2 \int_0^{\tau} T \exp[-(\tau - \bar{\tau})] \times d\bar{\tau} - 2 \int_{\tau}^{\infty} T \exp[-(\bar{\tau} - \tau)] \, d\bar{\tau} + 4T - 2T_r \exp(-\tau) \tag{10}$$

The dimensionless parameter δ_1 is given by

$$\delta_1 = \frac{3 \lambda k}{4 \sigma T_0^3} \tag{11}$$

and characterizes the ratio of the amount of heat conducted to the amount of heat radiated. A dimensionless time t has been introduced by defining

$$t = \frac{3 k \sigma T_0^3}{\rho c} t^* \tag{12}$$

The parameter multiplying t^* characterizes the ratio of the amount of heat radiated per unit time to the internal energy of the medium. The inverse of this parameter is the time required to radiate the entire internal energy of the gas at the rate determined by T_0.

Differential equation

If one now differentiates equation (10) twice, and substitutes the resulting equation back into equation (10) to eliminate the integral terms, one obtains

$$\delta_1 \frac{\partial^4 T}{\partial \tau^4} - (\delta_1 + 4) \frac{\partial^2 T}{\partial \tau^2} = \frac{\partial^3 T}{\partial \tau^2 \partial t} - \frac{\partial T}{\partial t} \tag{13}$$

If the radiative flux term had not been linearized, the resulting differential equation would be

$$\delta_1 \frac{\partial^4 T}{\partial \tau^4} - \delta_1 \frac{\partial^2 T}{\partial \tau^2} - \frac{\partial^2 (1 + T)^4}{\partial \tau^2} = \frac{\partial^3 T}{\partial \tau^2 \partial t} - \frac{\partial T}{\partial t}$$

where T no longer need be small.

The boundary conditions appropriate to the problem are

$$\begin{aligned} t = 0: &\quad T = 0 \\ x = 0: &\quad T = 0 \quad t < 0 \\ &\quad\;\; = T_s \quad t > 0 \\ &\quad q_r^+ = 0 \quad t < 0 \\ &\quad\;\; = 4T_r \quad t > 0 \\ x \to \infty: &\quad T \to 0 \\ &\quad q_r^+ \to 0 \end{aligned}$$

where q_r^+ is the radiative heat flux in the positive x-direction.

SOLUTION OF THE DIFFERENTIAL EQUATION

If a Laplace transformation in t is applied to equation (13), one obtains

$$\delta_1 \frac{d^4 \bar{T}}{d\tau^4} - (\delta_1 + 4 + p) \frac{d^2 \bar{T}}{d\tau^2} + p\bar{T} = 0 \tag{14}$$

where $\bar{T}$ is the Laplace transformation of T, i.e.

$$\bar{T} = \int_0^{\infty} \exp(-pt) T \, dt. \tag{15}$$

The solution of equation (14) can be written as

$$\bar{T} = A_1 \exp(\gamma_1 \tau) + A_2 \exp(\gamma_2 \tau) \tag{16}$$

where

$$\gamma_{1,2} = -\left[\frac{\delta_1 + 4 + p}{2\delta_1} \pm \frac{\sqrt{[(\delta_1 + 4 + p)^2 - 4\delta_1 p]}}{2\delta_1}\right]^{1/2} \tag{17}$$

By the use of the inversion integral, the solution of equation (13) can then be written as

$$2\pi i T = \int_{\Gamma} A_1 \exp(pt + \gamma_1 \tau) \, dp + \int_{\Gamma} A_2 \exp(pt + \gamma_2 \tau) \, dp \tag{18}$$

where Γ is the path to the right of all singularities such that $Re\, p =$ constant.

The integration constants A_1 and A_2 are determined by requiring that the solution, equation (16), satisfy the Laplace transform of the modified integro-differential equation, equation (10). The result is

$$A_1 = -\frac{T_s}{p}\frac{(1+\gamma_1)}{(\gamma_2-\gamma_1)} + \frac{T_r}{p}\frac{(1+\gamma_1)(1+\gamma_2)}{(\gamma_2-\gamma_1)} \tag{19}$$

$$A_2 = +\frac{T_s}{p}\frac{(1+\gamma_2)}{(\gamma_2-\gamma_1)} - \frac{T_r}{p}\frac{(1+\gamma_1)(1+\gamma_2)}{(\gamma_2-\gamma_1)} \tag{20}$$

Alternatively the solution, equation (18), could be obtained directly from equation (10) by Laplace transform methods. However, in general, the differential equation form is more informative especially in the non-linear case.

The integrals occurring in equation (18) can, of course, be evaluated by numerical means. However, in order to obtain and display the significant parameters of the problem, an analytic solution is preferable. An approximate analytic evaluation can be obtained for both small time ($t \ll 1$) and for long time ($t \gg 1$) as shown in the appendix. The general behavior for all time can then be inferred from these results.

EFFECT OF SURFACE TEMPERATURE

Approximation for small time

Since the problem has been linearized, the effects of surface temperature and incident external radiation can be separated. If there is no external radiation, i.e. $T_r = 0$, an approximate expression for the temperature valid when $t \ll 1$ is given by

$$T = T_s\left[\operatorname{erfc}\frac{\tau}{2\sqrt{(\delta_1 t)}} - 4\tau\left(\frac{t}{\delta_1}\right)^{1/2} i\operatorname{erfc}\frac{\tau}{2\sqrt{(\delta_1 t)}}\right] \tag{21}$$

Properties of the complimentary error function, erfc x, and its integrals, i^n erfc x, may be found in reference 6.

Since $\tau/2\sqrt{(\delta_1 t)} = x/2\sqrt{(\alpha t^*)}$ where α is the thermal diffusivity, the first term on the right-hand side of the above equation can be recognized as the usual conduction term.

The last term shows the effect of radiative transfer. As the radiative heat transfer becomes more significant, i.e. as δ_1 decreases, this term increases for small τ and decreases for large τ. Near the surface, the temperature increases since the emission of radiative energy, which is proportional to the temperature, is larger than the absorption of radiative energy, which is proportional to the amount of radiation present and, for small time, is negligible. Away from the surface, the temperature is increased because of the absorption of the increased radiative flux.

Asymptotic approximation

For long time, $t \gg 1$, the behavior of the solution is different depending on whether δ_1 is greater or less than 1. We first restrict our attention to the case $\delta_1 < 1$, i.e. when radiation is dominant.

For $t \gg 1$ and $\delta_1 < 1$, the temperature is given by

$$T = a_2 T_s \operatorname{erfc}\frac{\tau}{2\sqrt{(\delta_3 t)}} + a_1 T_s \exp(-\tau/a_2)\left[1 - \tfrac{1}{2}\operatorname{erfc}\left|\frac{\tau - at}{D\tau^{1/2}}\right|\right] \quad \text{for } t > \tau/a \tag{22}$$

$$T = a_2 T_s \operatorname{erfc}\frac{\tau}{2\sqrt{(\delta_3 t)}} + a_1 T_s \exp(-\tau/a_2)\operatorname{erfc}\left|\frac{\tau - at}{D\tau^{1/2}}\right| \quad \text{for } t < \tau/a \tag{23}$$

where

$$\delta_3 = \delta_1 + 4 \tag{24}$$

$$a_1 = \frac{(\delta_1+4)^{1/2} - \delta_1^{1/2}}{(\delta_1+4)^{1/2}} \tag{25}$$

$$a_2 = \frac{\delta_1^{1/2}}{(\delta_1+4)^{1/2}} \tag{26}$$

$$D = [2(1-\delta_1)\sqrt{\delta_1}]^{1/2} \tag{27}$$

$$a = \frac{\delta_1^{1/2}}{2}(\delta_1+4)^{3/2} \tag{28}$$

δ_3 is a parameter which characterizes the total diffusion of energy, both radiative and conductive. The first term on the right-hand sides of both equations (22) and (23) can therefore be considered as a diffusion term, including radiative and conductive diffusion, with an effective surface temperature $a_2 T_s$.

The last terms of equations (22) and (23) describe a wave-like motion with the following characteristics: The wave front propagates with a speed a. There is a change in temperature across the wave of $a_1 T_s \exp(-\tau/a_2)$ and hence the wave decays exponentially with a decay length defined by $\tau/a_2 = 1$, or $\tau = a_2$. The complimentary error function terms show the diffusive structure of the wave. A diffusion width can be defined as

$$\frac{\tau - at}{D\tau^{1/2}} = 1 \tag{29}$$

or

$$\tau - at = D\tau^{1/2} \tag{30}$$

The wave is caused by the following mechanism: For small time, high temperatures are restricted to small distances from the surface both because of the conductive diffusion and because of radiative emission as can be seen from equation (21). Hence the interior of the medium receives little energy either by radiation or conduction. As the high temperatures propagate inwards, the interior begins to receive energy, but once radiative energy is absorbed in a distance $1/k$, the mean free path for radiation, the effect is not felt instantaneously everywhere but propagates as a wave.

As $t \to \infty$, the temperature approaches the value

$$T = a_2 T_s + a_1 T_s \exp(-\tau/a_2) \tag{31}$$

and the interior temperature, in general, never reaches the temperature at the surface T_s.

A schematic diagram of the temperature field is shown in Fig. 1.

It may be noted that as radiation becomes dominant and $\delta_1 \to 0$, $a_2 \to 0$, $a_1 \to 1$, and the wave decays more rapidly. In the limit, the temperature is zero throughout the medium.

For $t \gg 1$ and $\delta_1 > 1$, the temperature is given by

$$T = a_2 T_s \operatorname{erfc} \frac{\tau}{2\sqrt{(\delta_3 t)}} + a_1 T_s \exp(-\tau/a_2) \tag{32}$$

No thermal wave as in the case $\delta_1 < 1$ is present. As $\delta_1 \to \infty$, $a_2 \to 1$, $a_1 \to 0$, $\delta_3 \to \delta_1$, and the solution approaches that given by pure conduction.

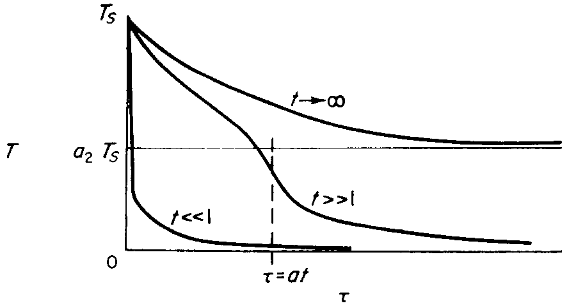

FIG. 1. Schematic diagram of temperature field for $T_r = 0$ and radiative transfer greater than conductive transfer, $\delta_1 = \frac{3\lambda k_r}{4\sigma T_0^3} < 1$. The temperature field is shown for small time ($t \ll 1$), large time ($t \gg 1$), and in the limit as $t \to \infty$.

EFFECT OF EXTERNAL RADIATION

Approximation for small time

If the surface temperature is zero but an external radiation field is present, the temperature within the medium is given for small time by

$$T = 2T_r \left[\left(\frac{t}{2\tau}\right)^{1/2} \exp(-\tau) I_1 [2\sqrt{(2\tau t)}] - 4ti^2 \operatorname{erfc} \frac{\tau}{2\sqrt{(\delta_1 t)}} \right] \tag{33}$$

where I_n is the modified Bessel function. Both terms on the right-hand side of the above expression are proportional to t for small time and hence are of less importance than either of the terms in equation (21). For $\tau = 0$, the above expression gives $T = 0$ as it must in order to satisfy the condition $T_s = 0$. As $\tau \to \infty$, the temperature again approaches zero.

Asymptotic approximation

Again let us consider first the case when $\delta_1 < 1$. For $t \gg 1$ and $\delta_1 < 1$, the temperature is given by

$$T = a_1 T_r \operatorname{erfc} \frac{\tau}{2\sqrt{(\delta_3 t)}} - a_1 T_r \exp(-\tau/a_2) \times \left[1 - \tfrac{1}{2} \operatorname{erfc} \left|\frac{\tau - at}{D\tau^{1/2}}\right|\right] \text{ for } t, > \tau/a \tag{34}$$

$$T = a_1 T_r \operatorname{erfc} \frac{\tau}{2\sqrt{(\delta_3 t)}} - \frac{a_1}{2} T_r \exp(-\tau/a_2) \times \operatorname{erfc} \left|\frac{\tau - at}{D\tau^{1/2}}\right| \text{ for } t < \tau/a \tag{35}$$

The first terms on the right-hand sides of the above equations show the diffusive character of the temperature field with a diffusivity δ_3 and an effective surface temperature a_1T_r. The last term describes a wave-like motion as in the previous case with the same attenuation and diffusion of the wave. However, the change in temperature across the wave is now $-a_1T_r \exp(-\tau/a_2)$ or a reversal in sign from the previous case.

The mechanism for the wave formation is effectively the reverse of that due to the surface temperature. For small time, the temperature reaches a maximum very near the surface. As time increases, this maximum temperature moves inward. However, the temperatures behind this maximum decrease more rapidly than those in front since the volume elements behind the point of maximum temperature are closer to the surface and lose energy more readily.

As $t \to \infty$, the temperature becomes

$$T = a_1T_r[1 - \exp(-\tau/a_2)] \quad (36)$$

A schematic diagram of the temperature field is shown in Fig. 2.

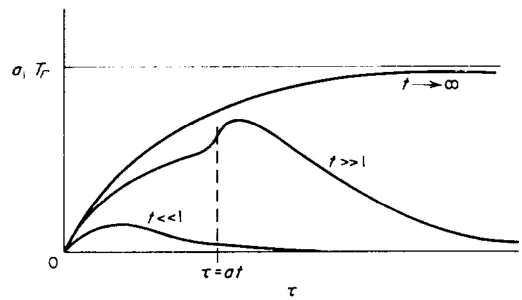

FIG. 2. Schematic diagram of temperature field for $T_s = 0$ and $\delta_1 < 1$. The temperature field is shown for small time ($t \ll 1$), large time ($t \gg 1$), and in the limit as $t \to \infty$.

For $t \gg 1$ and $\delta_1 > 1$, the temperature is

$$T = a_1T_r \operatorname{erfc} \frac{\tau}{2\sqrt{(\delta_3 t)}} - a_1T_r \exp(-\tau/a_2) \quad (37)$$

and is zero for $\tau = 0$ and as $\tau \to \infty$. The temperature has the same form as the asymptotic solution to the problem of the transient heat transfer by conduction alone in a semi-infinite medium with thermal diffusivity δ_3 and with heat being generated internally at a rate proportional to $a_1T_r \exp(-\tau/a_2)$ [6].

HEAT FLUX

If the temperature field is known, the heat flux can be found from equations (2) and (3). If we restrict our attention to the surface heat flux, these equations reduce to

$$q_c = -2\delta_1 \left(\frac{\partial T}{\partial \tau}\right)_{\tau=0} \quad (38)$$

$$q_r = -4 \int_0^\infty T(\tau) \exp[-\tau]\, d\tau + 4T_r \quad (39)$$

In the lowest order approximation keeping only the first term of equation (21), the heat flux for small time is

$$q_c = + \frac{2\delta_1 T_s}{\sqrt{(\pi\delta_1 t)}} \quad (40)$$

$$q_r = -4T_s[1 - \exp(\delta_1 t) \operatorname{erfc} \sqrt{(\delta_1 t)}] + 4T_r \quad (41)$$

In this approximation, the conductive heat flux is the same as in the pure conduction problem. The radiative heat flux is the sum of two components, an inward flux of $4T_r$ due to the external radiation, and an outward flux due to radiative emission from the medium itself.

As $t \to 0$, the radiative heat flux is

$$q_r = -8T_s \sqrt{\left(\frac{\delta_1 t}{\pi}\right)} + 4T_r \quad (42)$$

and increases as $\sqrt{t}$. Therefore, for small time, the conductive heat flux is dominant.

For $t \gg 1$ and for any value of δ_1, the surface heat flux in the first approximation is

$$q_c = 2\delta_1 \left[\frac{a_2T_s + a_1T_r}{\sqrt{(\pi\delta_3 t)}} - \frac{a_1}{a_2}(T_r - T_s)\right] \quad (43)$$

$$q_r = -4[a_2T_s + a_1T_r][1 - \exp(\delta_3 t) \operatorname{erfc} \sqrt{(\delta_3 t)}] + \frac{4a_1a_2}{a_2 + 1}(T_r - T_s) + 4T_r \quad (44)$$

As $t \to \infty$,

$$q_c = -\frac{8a_2}{a_2 + 1}(T_r - T_s) \quad (45)$$

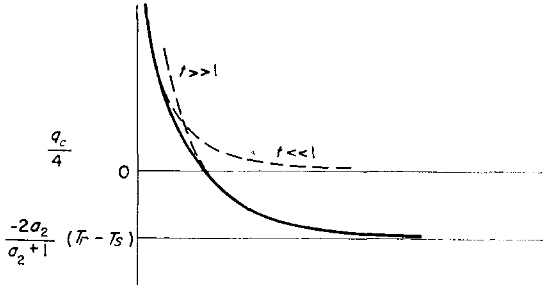

FIG. 3. Surface conductive heat flux as a function of time for $T_r > T_s$. Dashed lines show approximations for long and short time, while the solid line indicates the interpolated value for the conductive heat flux.

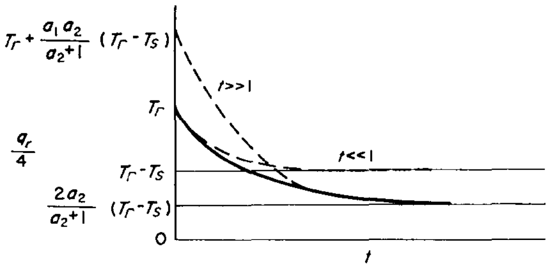

FIG. 4. Surface radiative heat flux as a function of time for $T_r > T_s$. Dashed lines show approximations for long and short time, while the solid line indicates the interpolated value for the radiative heat flux.

$$q_r = + \frac{8a_2}{a_2 + 1}(T_r - T_s) \tag{46}$$

and the net flux is then zero. The conductive and radiative surface heat fluxes are shown in Figs. 3 and 4.

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APPENDIX

Approximation for small time

An approximate evaluation of the integrals occurring in question (18) can be accomplished by substituting expansions for large p for the functions $A_{1,2}$ and $\gamma_{1,2}$. Since large p corresponds to high frequencies, this approximation is valid when the high frequency waves dominate, i.e. when t is small.

For large p,

$$\gamma_1 = -\left(\frac{p}{\delta_1}\right)^{1/2} + 0\left(\frac{1}{p^{1/2}}\right) \tag{A1}$$

$$\gamma_2 = -1 + \frac{2}{p} + 0\left(\frac{1}{p^2}\right) \tag{A2}$$

$$A_1 = \frac{T_s}{p} + \frac{2T_s}{p^2} + 0\left(\frac{1}{p^3}\right) \tag{A3}$$

$$A_2 = 0\left(\frac{1}{p^{5/2}}\right) \tag{A4}$$

If these expressions are substituted into equation (18), the integrations can be performed easily and equations (21) and (33) result.

Asymptotic approximation

For large time, the form of the integrals in equation (18) suggests evaluation either by the method of steepest descent or of stationary phase [7]. The method of stationary phase is more convenient for this problem. In this method, the dominant contributions to the integral come from the neighborhood of the stationary or saddle point and perhaps from any singularities enclosed by the contour path deformed to pass through the saddle point.

In order to determine the path of stationary phase, it is necessary to know the singular points of γ_1 and γ_2. γ_1 has branch points at

$$p_{1,2} = -4 + \delta_1 \pm 4i\delta_1^{1/2} \qquad (A5)$$

$$p_4 = \infty \qquad (A6)$$

and γ_2 has branch points at p_1, p_2, and $p_3 = 0$.

For the evaluation of T_2, the second integral in equation (18), we anticipate that the saddle point will be located near the origin. We then can approximate γ_2 and A_2 by expanding these functions for small p. The result is

$$\gamma_2 = -\left(\frac{p}{\delta_1 + 4}\right)^{1/2} + 0\,(p^{3/2}) \qquad (A7)$$

$$A_2 = \frac{a_2 T_s + a_1 T_r}{p} + 0\left(\frac{1}{p^2}\right) \qquad (A8)$$

Write the second integral in equation (18) as $\int A_2 \exp[f(p)t]\,dp$, where

$$f(p) = p + \gamma_2 \frac{\tau}{t} = -\frac{\tau}{\delta_3^{1/2} t} p^{1/2} + p + 0\,(p^{3/2}) \qquad (A9)$$

If we let $p = z^2$, then

$$f(z) = -\frac{\tau}{\delta_3^{1/2} t} z + z^2 + 0(z^3) \qquad (A10)$$

The saddle point is located at the point z_0 at which $f'(z_0) = 0$, or

$$z_0 = +\frac{\tau}{2\delta_3^{1/2} t} \qquad (A11)$$

In the vicinity of the saddle point,

$$f(z) = f(z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 \qquad (A12)$$

If the previous integration path Γ is now deformed so as to pass through the saddle point z_0 and so as to be a path of stationary phase, then, by the use of equations (A10) and (A12), an asymptotic value for T_2 is given by

$$T_2 = \frac{[a_2 T_s + a_1 T_r]}{\pi i} \exp[f(z_0)t] \int_{\Gamma'} \frac{\exp[tf''(z_0)(z - z_0)^2/2]}{z}\,dz \qquad (A13)$$

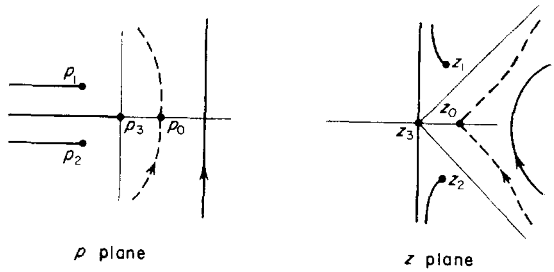

FIG. 5. Deformed integration path for the diffusion term T_2 of equation (18). The original path is shown by a solid line, while the path of stationary phase is indicated by a dashed line. The saddle point is located at p_0 in the p plane and z_0 in the z plane.

The path Γ', in the p and z planes, is as shown in Fig. 5. For small z, the path is a straight line at 45° to the real axis. Along this path the integral in equation (A13) can be written as

$$\int_{\Gamma'} \frac{\exp[a(z - z_0)^2]}{z}\,dz = \int_0^{\infty} \frac{\exp(ia\xi^2)}{\xi + q_1}\,d\xi + \int_{\infty}^{0} \frac{\exp(-ia\xi^2)}{\xi + q_2}\,d\xi \qquad (A14)$$

where

$$a = \frac{t}{2} f''(z_0)$$

$$q_1 = z_0 \exp(-i\pi/4)$$

$$q_2 = z_0 \exp(i\pi/4)$$

These integrals can then be integrated and lead to the result for T given as the first terms on the right-hand sides of equations (22) and (34).

For the evaluation of T_1, the first integral in equation (18), we again anticipate that the saddle point will be located near the origin. γ_1 and A_1 are then approximated by

$$\gamma_1 = -\frac{(\delta_1 + 4)^{1/2}}{\delta_1^{1/2}}\left[1 + \frac{2p}{(\delta_1 + 4)^2} + \frac{2(\delta_1 - 1)}{(\delta_1 + 4)^4} p^2 + 0\,(p^3)\right] \qquad (A15)$$

$$A_1 = -\frac{a_1(T_r - T_s)}{p} + 0\left(\frac{1}{p^2}\right) \qquad (A16)$$

If we write this integral as $\int A_1 \exp[f(p)t]\,\mathrm{d}p$, then

$$f(p) = -\frac{\tau}{a_2 t} + p\left(1 - \frac{\tau}{at}\right) - \frac{(\delta_1 \times 1)\tau}{a\delta_3^2 t}p^2 \tag{A17}$$

The saddle point is located at

$$p_0 = \frac{(at - \tau)}{\tau}\,\frac{\delta_3^2}{2(\delta_1 - 1)} \tag{A18}$$

and it follows that

$$T_1 = \frac{-a_1(T_r - T_s)}{2\pi i}\exp[f(p_0)t] \int_{\Gamma'} \frac{\exp[tf''(p_0)(p - p_0)^2/2]}{p}\,\mathrm{d}p \tag{A19}$$

The path of integration is shown in Fig. 6 and, for small p, is the same as the path of integration in the z plane, for small z, shown in Fig. 5. Since only the path near the saddle point is important and since the integrands of equations (A13) and (A14) are the same, the values of the integrals are the same.

In the present case, the saddle point may be located on either the positive or negative real axis. If the deformed path Γ' passing through the saddle point encloses the singular point $p = 0$, the contribution from this singular point must be included.

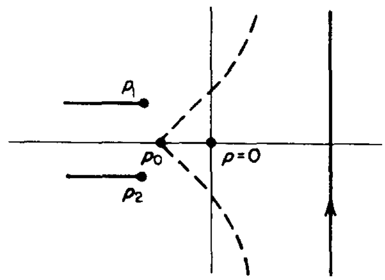

FIG. 6. Deformed integration path for the wave term T_1 of equation (18). The original path is shown by a solid line, the path of stationary phase by a dashed line. The saddle point is at p_0.

The results of these integrations are shown as the last terms of equations (22), (23), (34), and (35).

Résumé—Le transport d'énergie non-stationnaire par rayonnement et conduction à travers un milieu semi-infini a été étudié. On a employé une technique de substitution de noyau afin d'obtenir des solutions analytiques et, par suite, de montrer aisément les caractéristiques principales et les paramètres du problème. La température et le flux de chaleur sont donnés en fonction de la position et du temps et comparés avec la solution de la conduction pure.

Zusammenfassung—Die kurzzeitige Übertragung von Energie durch Strahlung und Leitung in einem halbunendlichen Medium wurde untersucht. Um analytische Lösungen zu erzielen und um die Grundzüge und Parameter des Problems völlig aufzuzeigen, wurden einzelne Terme durch Exponentialfunktionen ersetzt. Temperatur and Wärmestromdichte als Funktion von Ort und Zeit werden dargestellt und mit der Lösung für reine Leitung verglichen.

Аннотация—Изучался переходной процесс переноса энергии через полубесконечную среду излучением и теплопроводностью. Для того, чтобы получить аналитические решения и, следовательно, получить полную картину зависимости от основных переметров задачи, использовался метод замены ядра. Температура и тепловой поток, полученные как функции координаты и времени, сравниваются с решением для случая чистой теплопроводности.